Inequality with sum of ratios bisector to median.

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Let $l_a,\ l_b, l_c$ be lengths of bisectors of angles A, B and C in a triangle ABC and $m_a,\ m_b, m_c$ be lengths of correspondent medians. Prove that

$\dfrac{l_a}{m_a} + \dfrac{l_b}{m_b} + \dfrac{l_c}{m_c} > 1.$

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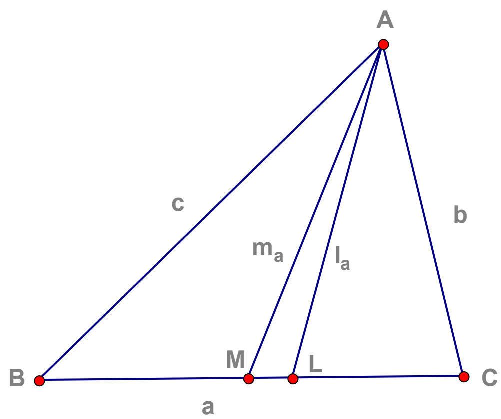

Let $\delta_a := ML$. Since $CL = \dfrac{ac}{b+c}$ and $BM = CM = \dfrac{a}{2}$ then

$\delta_a = \left|\dfrac{a}{2} - \dfrac{ac}{b+c}\right| = \dfrac{a|b-c|}{2(b+c)}$ and, therefore, we have $l_a + \delta_a \geq m_a \iff$

$l_a \geq m_a - \delta_a \iff \dfrac{l_a}{m_a} \geq 1 - \dfrac{\delta_a}{m_a} = 1 - \dfrac{a|b-c|}{2(b+c)m_a}.$

Assuming that $b \neq c$ (otherwise $\dfrac{l_a}{m_a} = 1$) we obtain

$2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 > 0 \iff$

$a^2(2(b^2+c^2) - a^2) - (b^2-c^2)^2 > 0 \iff 4a^2m_a^2 > (b-c)^2(b+c)^2 \iff$

$2am_a > |b-c|(b+c) \iff m_a > \dfrac{|b-c|(b+c)}{2a}$ then

$$\frac{a|b-c|}{2(b+c)m_a} < \frac{a|b-c|}{2(b+c)\cdot\dfrac{|b-c|(b+c)}{2a}} = \frac{a^2}{(b+c)^2}.$$

Hence, $\dfrac{l_a}{m_a} > 1 - \dfrac{a^2}{(b+c)^2} = \dfrac{4s(s-a)}{(b+c)^2}$ and this inequality obviously holds also if $b = c$.

Thus, $\sum \dfrac{l_a}{m_a} > 4s\sum \dfrac{s-a}{(b+c)^2}.$

Assuming that semiperimeter $s = 1$ (due homogeneity of the latter inequality) we obtain $x, y, z > 0, x+y+z = 1, a = 1-x, b = 1-y, c = 1-z, b+c = 1+x,$ $c+a = 1+y, a+b = 1+z$ and

$$4s\sum\frac{s-a}{(b+c)^2} = 4\sum\frac{x}{(1+x)^2} = \frac{4\sum x(1+y+z+yz)^2}{(2+p+q)^2} =$$

$$\frac{4\sum(xy^2z^2 + 2xy^2z + 2xyz^2 + xy^2 + xz^2 + 4xyz + 2xy + 2xz + x)}{(2+p+q)^2} =$$

$$\frac{4(pq + 4q + p - 3q + 12q + 4p + 1)}{(2 + p + q)^2} = \frac{20p + 52q + 4pq + 4}{(2 + p + q)^2} > 1$$

because $20p + 52q + 4pq + 4 - (2 + p + q)^2 = 16p + 48q - (p - q)^2 > p(16 - p) > 0.$